Approximate Unitary Equivalence of Homomorphisms from $\mathcal{O}_\infty$ *

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Abstract

We prove that if two homomorphisms from $\mathcal{O}_\infty$ to a purely infinite simple C^*-algebra have the same class in KK-theory, and if either both are unital or both are nonunital, then they are approximately unitarily equivalent. It follows that $\mathcal{O}_\infty$ is classifiable in the sense of Rørdam. In particular, Rørdam's classification theorem for direct limits of matrix algebras over even Cuntz algebras extends to direct limits involving both matrix algebras over even Cuntz algebras and corners of $\mathcal{O}_\infty$, for which the K_0-group can be an arbitrary countable abelian group with no even torsion.

0 Introduction

In [Rr1], Rørdam proved that a simple direct limit $A = \varinjlim A_n$ of finite direct sums of matrix algebras over even Cuntz algebras is classified up to isomorphism by the pair $(K_0(A), [1_A])$ consisting of its K_0-group together with the class of the identity. He furthermore proved that any pair (G, g), consisting of a countable abelian odd torsion group G and an element $g \in G$, can occur as $(K_0(A), [1_A])$ for such an algebra A. In this paper, we extend his classification by allowing, as additional summands in the algebras A_n, matrix algebras over the infinite Cuntz algebra $\mathcal{O}_\infty$ and arbitrary corners in it. One of the differences between $\mathcal{O}_n$ and $\mathcal{O}_\infty$ is that $K_0(\mathcal{O}_n)$ is finite while $K_0(\mathcal{O}_\infty)$ is infinite cyclic. This gives a class of algebras A for which $K_0(A)$ can be an arbitrary countable abelian group containing no even torsion, and in which $[1_A]$ can be an arbitrary element of this group.

Rørdam has defined in [Rr3] a "classifiable class" of purely infinite simple C^*-algebras, and has shown that algebras in this class can have arbitrary countable abelian groups as their K_0-groups (as well as possibly nontrivial K_1-groups). Algebras in his class are determined up to isomorphism by their K-theory together with the class of the identity. The new element in our work is that we use the natural choice of a C^*-algebra A satisfying $K_0(A) \cong \mathbf{Z}$ and $K_1(A) = 0$, namely $A = \mathcal{O}_\infty$, rather than the somewhat arbitrary construction of [Rr3]. Our results show that our algebras, in particular $\mathcal{O}_\infty$, are in fact in Rørdam's class. It follows that they are isomorphic to the C^*-algebras with the same K-theory constructed in [Rr3]. Perhaps more importantly, the C^*-algebra A, constructed according to the recipe in [Rr3] to satisfy $K_0(A) \cong \mathbf{Z}$ with generator $[1_A]$ and $K_1(A) = 0$, is actually isomorphic to $\mathcal{O}_\infty$.

Our main technical result is that if D is a purely infinite simple C^*-algebra, and if φ, $\psi : \mathcal{O}_\infty \to D$ are two unital homomorphisms with the same class in $KK^0(\mathcal{O}_\infty, D)$, then φ is approximately unitarily equivalent to ψ. Combined with

an easy existence result, this yields the statement that $\mathcal{O}_\infty$ is in Rørdam's class. (We actually use the simpler Definition 5.1 of [ER] rather than the definition in [Rr3].) The results described above then follow from [Rr3] and a variation of arguments from [Rr1].

In the first section, we establish terminology and notation, and prove several preliminary results, including approximate unitary equivalence of homomorphisms with trivial classes in KK-theory. In Section 2, we show that an arbitrary homomorphism from $\mathcal{O}_\infty$ to D is approximately absorbing in the sense of [LP] (see Definition 14). The last section contains the proof of the theorem for homomorphisms with arbitrary KK-classes, and the consequences discussed above.

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1 Preliminaries

We begin this section by recalling the definition and standard properties of the algebras $\mathcal{O}_\infty$ and E_n, and establishing notation for their standard generators. We then define approximately unitarily equivalent and approximately absorbing homomorphisms. Finally, we prove that if D is a purely infinite simple C^*-algebra, then up to approximate unitary equivalence there is only one unital homomorphism from $\mathcal{O}_\infty$ to D. This is an easy and important special case of our main technical theorem. We actually work with injective homomorphisms from E_n instead, but the two versions of the statement are equivalent.

1.1. Notation. Let $\mathcal{O}_n$ be the Cuntz algebra, and call its standard generators $s_1, \dots, s_n$. Thus the s_j are isometries satisfying $\sum_{j=1}^n s_j s_j^* = 1$.

Let E_n be the universal unital C^*-algebra on generators $t_1, \dots, t_n$ and relations stating that they are isometries with orthogonal ranges, but whose range projections do not necessarily sum to 1. Let J_n be the ideal generated by $1 - \sum_{j=1}^n t_j t_j^*$. It is known (Proposition 3.1 of [Cu1]) that $J_n \cong \mathcal{K}$, the algebra of compact operators on a separable infinite dimensional Hilbert space. Let $\pi_n : E_n \to \mathcal{O}_n$ be the quotient map. Thus $\pi_n(t_j) = s_j$, and we have a short exact sequence

$$0 \longrightarrow J_n \longrightarrow E_n \xrightarrow{\pi_n} \mathcal{O}_n \longrightarrow 0.$$

It is known (Proposition 3.9 of [Cu2]) that $K_0(E_n) \cong \mathbf{Z}$, generated by [1], and that $K_1(E_n) = 0$.

The algebra $\mathcal{O}_\infty$ can then be viewed as $\varinjlim E_n$, where the map $E_n \to E_{n+1}$ of the system sends t_j to t_j for $1 \le j \le n$. Accordingly, we will denote the generators of $\mathcal{O}_\infty$ by $t_1, t_2, \dots$, and identify E_n with the corresponding subalgebra

of $\mathcal{O}_\infty$. Recall (Corollary 3.11 of [Cu2]) that $K_0(\mathcal{O}_\infty) \cong \mathbf{Z}$, generated by [1], and that $K_1(\mathcal{O}_\infty) = 0$.

1.2 Lemma. Let A be either E_n or $\mathcal{O}_\infty$, and let D be any separable C^*-algebra. Then the Kasparov product $\alpha \mapsto [1_A] \times \alpha$, from $KK^0(A, D)$ to $K_0(D)$, is an isomorphism.

Proof: The Universal Coefficient Theorem ([RS], Theorem 1.18) shows that this is true for any C^*-algebra A in the bootstrap category $\mathcal{N}$ of [RS] such that $K_0(A) \cong \mathbf{Z}$ with generator [1] and $K_1(A) = 0$. Thus, we only need to show that our algebras A are in $\mathcal{N}$. Now $\mathcal{O}_n$ is stably isomorphic to a crossed product of an AF algebra by $\mathbf{Z}$ ([Cu1], 2.1) and so is in $\mathcal{N}$. Since E_n is an extension of $\mathcal{O}_n$ by $\mathcal{K}$, it too is in $\mathcal{N}$. Therefore $\mathcal{O}_\infty \cong \varinjlim E_n$ is also in $\mathcal{N}$. ∎

1.3 Lemma The standard defining relations for $\mathcal{O}_n$ and E_n are exactly stable in the sense of Loring [Lr]. That is:

(1) For each $\delta > 0$, let $\mathcal{O}_n(\delta)$ be the universal unital C^*-algebra on generators $s_{j,\delta}$ for $1 \leq j \leq n$ and relations

$$\|s_{j,\delta}^* s_{j,\delta} - 1\| \leq \delta \quad \text{and} \quad \|\sum_{k=1}^{n} s_{k,\delta} s_{k,\delta}^* - 1\| \leq \delta,$$

and let $\kappa_\delta : \mathcal{O}_n(\delta) \to \mathcal{O}_n$ be the homomorphism given by $\kappa_\delta(s_{j,\delta}) = s_j$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that there is a homomorphism $\varphi_\delta : \mathcal{O}_n \to \mathcal{O}_n(\delta)$ satisfying $\kappa_\delta \circ \varphi_\delta = \mathrm{id}_{\mathcal{O}_n}$ and $\|\varphi_\delta(s_j) - s_{j,\delta}\| < \varepsilon$ for all j.

(2) For each $\delta > 0$, let $E_n(\delta)$ be the universal unital C^*-algebra on generators $t_{j,\delta}$ for $1 \leq j \leq n$ and relations

$$\|t_{j,\delta}^* t_{j,\delta} - 1\| \leq \delta \quad \text{and} \quad \|(t_{j,\delta} t_{j,\delta}^*)(t_{k,\delta} t_{k,\delta}^*)\| \leq \delta$$

for $j \neq k$, and let $\kappa_\delta : E_n(\delta) \to E_n$ be the homomorphism given by $\kappa_\delta(t_{j,\delta}) = t_j$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that there is a homomorphism $\varphi_\delta : E_n \to E_n(\delta)$ satisfying $\kappa_\delta \circ \varphi_\delta = \mathrm{id}_{E_n}$ and $\|\varphi_\delta(t_j) - t_{j,\delta}\| < \varepsilon$ for all j.

Proof: Part (1) has already been observed to follow from results in the literature; see the proof of Lemma 2.1 of [LP]. In any case, it can be proved directly by the same argument as for (2). We therefore only do (2). Since the methods are standard, we will be somewhat sketchy. In particular, we will successively form the $-1/2$ powers of elements $x_0, \ldots, x_{n-1}$ of $E_n(\delta)$ (namely the elements $((1 - q_k)t_{k+1,\delta})^*(1 - q_k)t_{k+1,\delta}$ below). Each will be a perturbation of $t_{k+1,\delta}^* t_{k+1,\delta}$ satisfying $\|x_k - 1\|$ small for δ small enough, but for fixed δ the norms $\|x_k - 1\|$ will grow fairly rapidly with k. The number δ must be small enough that all of the $\|x_k - 1\|$ are small.

Define $w_1 \in E_n(\delta)$ by $w_1 = t_{1,\delta}(t_{1,\delta}^* t_{1,\delta})^{-1/2}$. Then w_1 is an isometry, $\|w_1 - t_{1,\delta}\|$ is small, and $\kappa_\delta(w_1) = t_1$.

Now suppose we have constructed isometries $w_1, \dots, w_k$ (with $k < n$) such that $\|w_j - t_{j,\delta}\|$ is small, $\kappa_\delta(w_j) = t_j$, and $w_1 w_1^*, \dots, w_k w_k^*$ are mutually orthogonal projections. Let $q_k = w_1 w_1^* + \cdots + w_k w_k^*$. Since $w_j w_j^*$ is close to $t_{j,\delta} t_{j,\delta}^*$, it follows from the relations for $E_n(\delta)$ that $\|q_k t_{k+1,\delta} t_{k+1,\delta}^* q_k\|$ is small. Therefore so is $\|t_{k+1,\delta}^* q_k t_{k+1,\delta}\|$, whence $(1 - q_k) t_{k+1,\delta}$ is close to $t_{k+1,\delta}$. Now set

$$w_{k+1} = (1 - q_k) t_{k+1,\delta} [((1 - q_k) t_{k+1,\delta})^* (1 - q_k) t_{k+1,\delta}]^{-1/2}.$$

It follows that w_{k+1} is an isometry which is close to $t_{k+1,\delta}$, has range orthogonal to q_k, and satisfies $\kappa_\delta(w_{k+1}) = t_{k+1}$.

Since only finitely many steps are required, if δ is small enough we obtain by induction isometries $w_1, \dots, w_n \in E_n(\delta)$ with orthogonal ranges such that $\kappa_\delta(w_j) = t_j$. If δ is sufficiently small, then we will also have $\|w_j - t_{j,\delta}\| < \varepsilon$. We then define $\varphi_\delta(t_j) = w_j$. ∎

The following three definitions are essentially the same as corresponding definitions in [LP].

1.4 Definition. Let A and B be C^*-algebras, let F be a finite subset of A, and let φ and ψ be two homomorphisms from A to B. We say that φ and ψ are *approximately unitarily equivalent to within* ε, with respect to F, if there is a unitary $v \in \widetilde{B}$ such that

$$\|\varphi(f) - v\psi(f)v^*\| < \varepsilon$$

for all $f \in F$. We abbreviate this as

$$\varphi \overset{\varepsilon}{\sim} \psi$$

with respect to F. When the set F is understood, we omit mention of it.

We further say that φ and ψ are *approximately unitarily equivalent* if for every finite $F \subset A$ and $\varepsilon > 0$, we have $\varphi \overset{\varepsilon}{\sim} \psi$ with respect to F.

We make two comments on this definition. First, if A has a finite generating set G, and if $\varphi \overset{\varepsilon}{\sim} \psi$ with respect to G for all $\varepsilon > 0$, then φ is approximately unitarily equivalent to ψ. Second, if $F_1 \subset F_2 \subset \cdots$ are finite subsets of A such that $G = \bigcup_{n=1}^{\infty} F_n$ generates A, and if $\varphi \overset{\varepsilon}{\sim} \psi$ with respect to F_n for all n and all $\varepsilon > 0$, then again φ is approximately unitarily equivalent to ψ.

1.5. Definition Let A be any unital C^*-algebra, and let D be a purely infinite simple C^*-algebra. Let $\varphi, \psi : A \to D$ be two homomorphisms, and assume that $\varphi(1) \neq 0$ and $[\psi(1)] = 0$ in $K_0(D)$. We define a homomorphism $\varphi \widetilde{\oplus} \psi : A \to D$, well defined up to unitary equivalence, by the following construction. Choose a projection $q \in D$ such that $0 < q < \varphi(1)$ and $[q] = 0$. Since D is purely infinite and simple, there are partial isometries v and w such that $vv^* = \varphi(1) - q$, $v^*v = \varphi(1)$, $ww^* = q$, and $w^*w = \psi(1)$. Now define $(\varphi \widetilde{\oplus} \psi)(a) = v\varphi(a)v^* + w\psi(a)w^*$ for $a \in A$.

1.6 Definition Let D be a purely infinite simple C^*-algebra, let A be $\mathcal{O}_n$, $\mathcal{O}_\infty$, E_n, or a finite matrix algebra over one of these, and let $\varphi : A \to D$ be a homomorphism. Then φ is *approximately absorbing* if for every $\psi : A \to D$ such that $[\psi] = 0$ in $KK^0(A, D)$, the homomorphisms φ and $\varphi \widetilde{\oplus} \psi$ are approximately unitarily equivalent.

The following proposition is stated in terms of E_n, but it immediately implies the same statement about homomorphisms from $\mathcal{O}_\infty$.

1.7 Proposition Let D be a unital purely infinite simple C^*-algebra. Let φ, $\psi : E_n \to D$ be two injective homomorphisms, either both unital or both nonunital. If $[\varphi] = [\psi] = 0$ in $KK^0(E_n, D)$, then φ and ψ are approximately unitarily equivalent.

Proof: We will show that, for any $\varepsilon > 0$ and any integer n, there is a unitary $w \in D$ such that

$$\|w^* \varphi(t_j) w - \psi(t_j)\| < \varepsilon$$

for $j = 1, 2, \ldots, n$. Since $[\varphi(1)] = [\psi(1)]$, by applying an inner automorphism, we may assume that $\varphi(1) = \psi(1)$. Furthermore, replacing D by $\varphi(1) D \varphi(1)$, we may assume that $\varphi(1) = 1$. For each j, we have

$$[\varphi(t_j t_j^*)] = [\psi(t_j t_j^*)]$$

in $K_0(D)$. Since $\{\varphi(t_j t_j^*)\}$ and $\{\psi(t_j t_j^*)\}$ are both sequences of mutually orthogonal nonzero projections in D, there is a unitary $u \in D$ such that

$$u^* \varphi(t_j t_j^*) u = \psi(t_j t_j^*)$$

for $j = 1, 2, \ldots, n$. Replacing φ by $u^* \varphi(-) u$, we may assume that

$$\varphi(t_j t_j^*) = \psi(t_j t_j^*)$$

for $j = 1, 2, \ldots, n$. Let p_j be this common value, and set $q = 1 - \sum_{j=1}^n p_j$. Since $[\varphi] = [\psi] = 0$, we have

$$[1] = \left[\sum_{j=1}^n p_j \right] = [q] = 0$$

in $K_0(D)$. Therefore there are $g_1, g_2, \ldots, g_n \in D$ such that

$$g_j^* g_j = 1 \quad \text{and} \quad \sum_{j=1}^n g_j g_j^* = q.$$

Define $\tilde{\varphi} : \mathcal{O}_{2n} \to D$ by

$$\tilde{\varphi}(s_j) = \varphi(t_j) \quad \text{and} \quad \tilde{\varphi}(s_{n+j}) = g_j$$

for $j = 1, 2, \ldots, n$.

Let $v_0 = \sum_{j=1}^{n} \psi(t_j)\varphi(t_j)^*$. Then v_0 is a unitary in $(1-q)D(1-q)$. Since D is purely infinite and simple, there is a unitary $v_1 \in qDq$ such that $[v_1] = [v_0^*]$ in $K_1(D)$. Set $w_j = v_1 g_j$. Then $\sum_{j=1}^{n} w_j g_j^* = v_1$. Define $\tilde{\psi} : \mathcal{O}_{2n} \to D$ by

$$\tilde{\psi}(s_j) = \psi(t_j) \quad \text{and} \quad \tilde{\psi}(s_{n+j}) = w_j$$

for $j = 1, 2, \ldots, n$. Then

$$\sum_{j=1}^{2n} \tilde{\psi}(s_j)\tilde{\varphi}(s_j)^* = v_0 + v_1 \in U_0(D).$$

By Theorem 3.6 of [Rr1], there exists a unitary $w \in D$ such that

$$\|w^* \tilde{\varphi}(s_j) w - \tilde{\psi}(s_j)\| < \varepsilon$$

for $j = 1, 2, \ldots, 2n$. In particular,

$$\|w^* \varphi(t_j) w - \psi(t_j)\| < \varepsilon$$

for $j = 1, 2, \ldots, n$. ∎

2 Approximate absorption

In this section, we prove that if D is a purely infinite simple C^*-algebra, then any injective homomorphism from E_n to D is approximately absorbing. (This result immediately implies a corresponding result for homomorphisms from $\mathcal{O}_\infty$ to D.) As we will see in the next section, approximate unitary equivalence of homomorphisms with the same class in KK-theory will follow easily.

The technical part of this section is the construction, for even n, of an approximately central projection in E_n whose K_0-class is zero. We first construct a copy of $\mathcal{O}_n \oplus E_n$ inside E_n such that the class of $(1, 0)$ in $K_0(E_n)$ is trivial. Then we use Rørdam's work for even Cuntz algebras, and Voiculescu's Theorem, to move this copy so that the image of (s_j, t_j) is close to t_j.

We mention another approach, not used here since it takes longer to write. We really only need an approximately central projection in $\mathcal{O}_\infty$. The inclusion of E_n in $\mathcal{O}_\infty$ can be extended to an inclusion of a suitable Cuntz-Krieger algebra $\mathcal{O}_A$ in $\mathcal{O}_\infty$, in such a way that $K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_\infty)$ is an isomorphism. We have shown that the shift on a Cuntz-Krieger algebra (see Section 4 of [Rr2]) satisfies a version of the approximate Rokhlin property of [BEK]. This yields approximately central projections, which with a little extra work can be chosen to be trivial in K_0.

2.1 Lemma Let p, $q \in E_n \setminus J_n$ be two projections. If $[p] = [q]$ in $K_0(E_n)$, then p is Murray-von Neumann equivalent to q.

Proof: Let $\mathcal{P}$ be the set of projections in E_n which are not in J_n. We show that $\mathcal{P}$ satisfies the conditions (Π_1)–(Π_4) before 1.3 of [Cu2]. The conclusion will then follow from Theorem 1.4 of [Cu2].

Conditions (Π_1) (the sum of two orthogonal projections in $\mathcal{P}$ is again in $\mathcal{P}$), (Π_2) ($\mathcal{P}$ is closed under Murray-von Neumann equivalence), and (Π_4) ($p \in \mathcal{P}$ and $p \leq q$ imply $q \in \mathcal{P}$) are all obvious. We thus prove (Π_3). That is, we must show that if $p, q \in E_n \setminus J_n$ are projections, then there is a projection $p' \in E_n \setminus J_n$ such that $p' < q$, p' is Murray-von Neumann equivalent to p, and $q - p' \in E_n \setminus J_n$. Now note that any projection is Murray-von Neumann equivalent to a subprojection of $t_1 t_1^*$, since this projection is Murray-von Neumann equivalent to 1. Therefore it suffices to prove the condition with $p = t_1 t_1^*$.

Since $\mathcal{O}_n$ is purely infinite and simple, we can find a projection $\bar{f} < \pi_n(q)$ and a unitary $\bar{u} \in \mathcal{O}_n$ such that $\bar{u}\pi_n(p)\bar{u}^* = \bar{f}$. Since $U(\mathcal{O}_n)$ is connected, there is $u \in U(E_n)$ such that $\pi_n(u) = \bar{u}$. Let $f = upu^*$; then $\pi_n(f) < \pi_n(q)$.

We now want to find $e \in (1-q)J_n(1-q)$ and a unitary v such that $q + e > vfv^*$. If $1 - q \in J_n$, we can take $e = 1 - q$ and $v = 1$. So assume $1 - q \notin J_n$. Then $(1-q)J_n(1-q)$ is isomorphic to $\mathcal{K}$, and so has an approximate identity $\{e_k\}$ consisting of projections. Since $f - qf \in J_n$, we have

$$(1 - q - e_k)f = (1 - q - e_k)(f - qf) = f - qf - e_k(f - qf) \to 0$$

as $n \to \infty$. Therefore $(q + e_k)f \to f$. It follows, for large enough k, that f is unitarily equivalent to a subprojection g of $q + e_k$, via a unitary that is close to 1. Then $\pi_n(q + e_k - g)$ is close to $\pi_n(q) - \bar{f}$, and in particular is not zero. Thus $q + e_k - g \notin J_n$.

It now suffices to show that $q + e_k$ is Murray-von Neumann equivalent to a (not necessarily proper) subprojection of q. Note that e_k is a finite sum of minimal projections in $(1-q)J_n(1-q)$. Since this algebra is isomorphic to $\mathcal{K}$, all minimal projections are equivalent, and it suffices to show that there exists a nonzero projection $e_0 \in (1-q)J_n(1-q)$ such that $q + e_0$ is Murray-von Neumann equivalent to a subprojection of q. Since $J_n \cong \mathcal{K}$, it is equivalent to show that there is a nonzero projection $e \in qJ_nq$ such that q is Murray-von Neumann equivalent to a subprojection of $q - e$.

We now claim that partial isometries in the quotient $\pi_n(q)\mathcal{O}_n\pi_n(q)$ lift to partial isometries in qE_nq. This follows from Corollary 2.12, the proof of Lemma 2.8, and Remark 2.9 in [Zh], since qJ_nq, being isomorphic to $\mathcal{K}$, has real rank zero and trivial K_1-group. (This is also known to operator theorists. Further see the proof of Lemma 2.6 of [Ell].)

It follows from this claim that there is a partial isometry $s_0 \in qE_nq$ such that $\pi_n(s_0)$ is a proper isometry in $\pi_n(q)\mathcal{O}_n\pi_n(q)$. Then $q - s_0^* s_0$ is a finite rank projection in $qJ_nq \cong \mathcal{K}$. Furthermore, $q - s_0 s_0^* \notin J_n$, but J_n is an essential ideal, so $(q - s_0 s_0^*)J_n(q - s_0 s_0^*)$ contains projections of arbitrarily large rank.

Therefore s_0 can be extended to an isometry s in qE_nq. For the same reason as for s_0, there are projections in $(q-ss^*)J_n(q-ss^*)$ of arbitrarily large rank. The existence of the required projection e now follows. This completes the proof. ∎

2.2. Lemma Let n be an even number. Then for any $\varepsilon > 0$ there exist a projection $f \in E_n \setminus J_n$ and partial isometries

$$v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)} \in fE_nf \quad \text{and} \quad v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)} \in (1-f)E_n(1-f)$$

such that $[f] = 0$ in $K_0(E_n)$,

$$(v_j^{(1)})^* v_j^{(1)} = f, \quad \sum_{k=1}^{n} v_k^{(1)} (v_k^{(1)})^* = f,$$

$$(v_j^{(2)})^* v_j^{(2)} = 1-f, \quad \sum_{k=1}^{n} v_k^{(2)} (v_k^{(2)})^* < 1-f,$$

and $\|v_j^{(1)} + v_j^{(2)} - t_j\| < \varepsilon$ for $j = 1, \ldots, n$.

Proof: Let $e_j = t_j(1 - t_1t_1^*)t_j^*$. Then $e_1, e_2, \ldots, e_n$ are n mutually orthogonal projections in $E_n \setminus J_n$ whose classes are 0 in $K_0(E_n)$. Let $e = \sum_{j=1}^n e_j$. By Lemma 2.1, there are $z_j^{(1)} \in eE_ne$ such that

$$(z_j^{(1)})^* z_j^{(1)} = e \quad \text{and} \quad z_j^{(1)} (z_j^{(1)})^* = e_j.$$

Similarly, there is $w \in E_n$ such that

$$w^*w = 1 \quad \text{and} \quad ww^* = 1 - e.$$

Define $z_j^{(2)} = wt_jw^*$ and $z_j = z_j^{(1)} + z_j^{(2)}$ for $j = 1, 2, \ldots, n$. Then define $\psi : E_n \to E_n$ by $\psi(t_j) = z_j$. Note that $[\psi] = 1$ in $KK^0(E_n, E_n)$ by Lemma 1.2. Set $z_j = \psi(s_j)$ and note that $(1-e) - \sum_{j=1}^n z_j^{(2)} (z_j^{(2)})^*$ is a rank one projection in $(1-e)J_n(1-e)$ (which is isomorphic to $\mathcal{K}$). It follows that the elements $\pi_n(z_j) \in \mathcal{O}_n$ satisfy

$$\pi_n(z_j)^* \pi_n(z_j) = 1 \quad \text{and} \quad \sum_{j=1}^{n} \pi_n(z_j) \pi_n(z_j)^* = 1.$$

We now have two homomorphisms from $\mathcal{O}_n$ to $\mathcal{O}_n$, given by $s_j \mapsto \pi_n(z_j)$ and $s_j \mapsto \pi_n(t_j) = s_j$. They both induce the identity map on $K_0(\mathcal{O}_n)$. Since $K_1(\mathcal{O}_n) = 0$, the Universal Coefficient Theorem ([RS], Theorem 1.18) implies that they have the same class in KK-theory. Let $\delta > 0$ and $\eta > 0$ be small numbers (to be chosen below; δ will depend on η). By Theorem 3.6 of [Rr1] there is a unitary $v \in \mathcal{O}_n$ such that

$$\|v^* \pi_n(z_j) v - \pi_n(t_j)\| < \delta/2$$

for $j = 1, 2, \ldots, n$. Since $U(\mathcal{O}_n)$ is connected, there is a unitary $u \in E_n$ such that $\pi_n(u) = v$. Then there are $a_j \in J_n$ such that

$$\|u^* z_j u - (t_j + a_j)\| < \delta.$$

for $j = 1, 2, \ldots, n$.

If δ is sufficiently small, then by Lemma 1.3(2) there are isometries $t_j' \in E_n$ with orthogonal ranges such that $\pi_n(t_j') = \pi_n(t_j)$ and

$$\|t_j + a_j - t_j'\| < \eta$$

for $j = 1, \ldots, n$. It follows that

$$\|t_j' - u^* z_j u\| < \eta + \delta.$$

Let H be an infinite dimensional separable Hilbert space. Recall that $J_n \cong \mathcal{K}(H)$. A standard result in representation theory yields a (unique) representation $\rho : E_n \to B(H)$ which extends this isomorphism. Clearly ρ is irreducible, and it is faithful because J_n is an essential ideal in E_n. Define a second representation $\sigma : E_n \to B(H)$ by $\sigma(t_j) = t_j'$. We are going to use Voiculescu's Theorem, as stated in Arveson's paper [Ar], to prove that ρ and σ are approximately unitarily equivalent.

Note that

$$\left\| 1 - \sum_{j=1}^{n} t_j'(t_j')^* - u^* \left(1 - \sum_{j=1}^{n} z_j z_j^* \right) u \right\| < n(\eta + \delta),$$

and recall that $1 - \sum_{j=1}^{n} z_j z_j^*$ is a rank one projection in J_n. If $\eta + \delta$ is small enough, it follows that

$$\sigma \left(1 - \sum_{j=1}^{n} t_j t_j^* \right) = \rho \left(1 - \sum_{j=1}^{n} t_j'(t_j')^* \right)$$

is a rank one projection in $\mathcal{K}(H)$. Since it is not zero, σ is a faithful representation of E_n. Let $H_0 = \overline{\sigma(J_n)H}$, the essential subspace of $\sigma|_{J_n}$. Note that it is a reducing subspace for σ. Since $\sigma(1 - \sum_{j=1}^{n} t_j t_j^*)$ has rank one, we conclude that $(\sigma|_{J_n})|_{H_0}$ is irreducible, and hence unitarily equivalent to $\rho|_{J_n}$. Standard results in representation theory now imply that $\sigma(-)|_{H_0}$ is unitarily equivalent to ρ.

We have now verified the hypotheses of Theorem 5(iii) of [Ar]: ρ and σ have the same kernel (namely $\{0\}$), their compositions with the quotient map from $B(H)$ to the Calkin algebra have the same kernel (namely J_n), and the essential parts are unitarily equivalent. Since $\sigma(t_j) = \rho(t_j')$, that theorem yields a unitary $W \in B(H)$ such that

$$\|W^* \rho(t_j) W - \rho(t_j')\| < \eta \quad \text{and} \quad W^* \rho(t_j) W - \rho(t_j') \in \mathcal{K}(H)$$

for $j = 1, 2, \ldots, n$. Since $t_j - t'_j \in J_n$ and $W^*\rho(t_j)W - \rho(t'_j) \in \mathcal{K}(H)$, we obtain $\rho(t_j) - W^*\rho(t_j)W \in \mathcal{K}(H)$, whence $W^*\rho(t_j)W \in \rho(E_n)$. Furthermore, the C^*-subalgebra generated by $\{W^*\rho(t_j)W : j = 1, 2, \ldots, n\}$ contains $W^*\mathcal{K}(H)W = \mathcal{K}(H)$, and therefore also contains $\rho(t_j)$. This set thus generates $\rho(E_n)$. Since ρ is faithful, it follows that there is an automorphism $\alpha : E_n \to E_n$ such that $\rho(\alpha(t_j)) = W^*\rho(t_j)W$ for $j = 1, 2, \ldots, n$. We can then combine earlier estimates to obtain

$$\|\alpha(t_j) - u^* z_j u\| < 2\eta + \delta$$

for $j = 1, 2, \ldots, n$. Set

$$f = \alpha^{-1}(u^* e u) \quad \text{and} \quad v_j^{(i)} = \alpha^{-1}(u^* z_j^{(i)} u)$$

for $i = 1, 2$ and $j = 1, \ldots, n$. Note that $[f] = 0$ in $K_0(E_n)$. If $2\eta + \delta \leq \varepsilon$, then these are the desired elements. ∎

2.3 Proposition Let D be a purely infinite simple C^*-algebra and let $\varphi : E_n \to D$ be a monomorphism. Then φ is approximately absorbing.

Proof: Replacing D by $\varphi(1)D\varphi(1)$, we may assume that D and φ are unital. Since $q = \varphi(1 - \sum_{j=1}^{n} t_j t_j^*)$ is nonzero, there are $n + 1$ mutually orthogonal nonzero projections

$$p_{n+1}, p_{n+2}, \ldots, p_{2n}, e \in qDq$$

and isometries

$$\tilde{t}_{n+1}, \tilde{t}_{n+2}, \ldots, \tilde{t}_{2n} \in D$$

such that $\tilde{t}_j \tilde{t}_j^* = p_j$ for $j = n+1, n+2, \ldots, 2n$. Now let $A \subset D$ be the C^*-subalgebra generated by $\varphi(t_j)$ for $j = 1, 2, \ldots, n$ and $\tilde{t}_j$ for $j = n+1, n+2, \ldots, 2n$. Then A is isomorphic to E_{2n}. By Lemma 2.2, for any $\varepsilon > 0$ there is a projection $f \in A$ and unital homomorphisms $\psi_1 : \mathcal{O}_{2n} \to fDf$ and $\psi_2 : E_{2n} \to (1-f)D(1-f)$ such that $[f] = 0$ in $K_0(A)$ (and hence in $K_0(D)$), and

$$\|\varphi(t_j) - (\psi_1(s_j) + \psi_2(t_j))\| < \varepsilon/3$$

for $1 \leq j \leq n$ and

$$\|\tilde{t}_j - (\psi_1(s_j) + \psi_2(t_j))\| < \varepsilon/3$$

for $n + 1 \leq j \leq 2n$. Define φ_1, $\varphi_2 : E_n \to D$ by $\varphi_1(t_j) = \psi_1(s_j)$ and $\varphi_2(t_j) = \psi_2(t_j)$ for $j = 1, \ldots, n$. Note that $[\varphi_1] = 0$ in $KK^0(E_n, D)$ by Lemma 1.2.

Now let $\varphi_0 : E_n \to D$ be any homomorphism with $[\varphi_0] = 0$ in $KK^0(E_n, D)$. Without loss of generality, we may assume $\varphi_0(1) \leq \varphi_1(1)$. Then $\varphi_1 \stackrel{\varepsilon/3}{\sim} \varphi_1 \widetilde{\oplus} \varphi_0$ by Proposition 1.7. Therefore

$$\varphi \stackrel{\varepsilon/3}{\sim} \varphi_1 + \varphi_2 \stackrel{\varepsilon/3}{\sim} (\varphi_1 \widetilde{\oplus} \varphi_0) + \varphi_2 \stackrel{\varepsilon/3}{\sim} \varphi \widetilde{\oplus} \varphi_0,$$

so $\varphi \stackrel{\varepsilon}{\sim} \varphi \widetilde{\oplus} \varphi_0$ as desired. ∎

3 Classification and direct limits

We start this section by proving our main technical theorem, that homomorphisms from $\mathcal{O}_\infty$ to purely infinite simple C^*-algebras with the same KK-classes are approximately unitarily equivalent. This implies that $\mathcal{O}_\infty$ is in the "classifiable class" $\mathcal{C}$ of [ER] (a slight modification of the class in [Rr3]). As discussed in the introduction, we then obtain classification theorems for direct limits involving even Cuntz algebras and corners of $\mathcal{O}_\infty$, for which the K_0-groups can have elements of infinite order. As an interesting corollary, we prove that if $p \in \mathcal{O}_\infty$ is a projection such that $[p] = -[1]$ in $K_0(\mathcal{O}_\infty)$, then $p\mathcal{O}_\infty p \cong \mathcal{O}_\infty$.

3.1. Definition Let $\varphi : \mathcal{O}_\infty \to D$ be a homomorphism. Let $p_j = \varphi(t_j t_j^*)$ for $j = 1, 2, \ldots$. Define $\bar{\varphi} : \mathcal{O}_\infty \to (1 - p_1 - p_2)D(1 - p_1 - p_2)$ by

$$\bar{\varphi}(t_j) = \varphi(t_{j+2})(1 - p_1 - p_2)$$

for $j = 1, 2, \ldots$.

3.2 Lemma Let φ and $\bar{\varphi}$ be as in the previous definition, and set

$$D_0 = [1 \oplus (1 - p_1 - p_2)]M_2(D)[1 \oplus (1 - p_1 - p_2)].$$

Then the direct sum

$$\varphi \oplus \bar{\varphi} : \mathcal{O}_\infty \to D_0$$

satisfies $[\varphi \oplus \tilde{\varphi}] = 0$ in $KK^0(\mathcal{O}_\infty, D_0)$.

Proof: Clearly $[(\varphi \oplus \tilde{\varphi})(1)] = 0$ in $K_0(D_0)$. The result is now immediate from Lemma 1.2. ∎

3.3. Theorem Let D be a purely infinite simple C^*-algebra, and let φ, ψ : $\mathcal{O}_\infty \to D$ be homomorphisms such that $[\varphi] = [\psi]$ in $KK^0(\mathcal{O}_\infty, D)$ (equivalently, $[\varphi(1)] = [\psi(1)]$ in $K_0(D)$). If φ and ψ are both unital or both nonunital, then φ and ψ are approximately unitarily equivalent.

Proof: We first reduce to the unital case. If both homomorphisms are nonunital, then the hypotheses imply that $\varphi(1)$ is unitarily equivalent to $\psi(1)$. Conjugating ψ by a suitable unitary, we may thus assume that $\varphi(1) = \psi(1)$. Now replace D by $\varphi(1)D\varphi(1)$.

We now follow the notation of Definition 3.1. Clearly there is a partial isometry $W \in M_3(O_\infty)$ such that

$$W^*W = 1 \oplus (1 - p_1 - p_2) \oplus 1 \quad \text{and} \quad WW^* = 1 \oplus 0 \oplus 0.$$

Let $G = \{t_1, t_2, \ldots\}$ be the standard (infinite) set of generators of $\mathcal{O}_\infty$, and let $G_k = \{t_1, t_2, \ldots, t_k\}$ be the set consisting of the first k of them. Note that G_k generates the canonical copy of E_k in $\mathcal{O}_\infty$. Both $[\varphi \oplus \bar{\varphi}]$ and $[\bar{\varphi} \oplus \psi]$ are zero

in $KK^0(\mathcal{O}_\infty, D)$, so Lemma 1.2 implies that $[(\varphi \oplus \bar{\varphi})|_{E_k}] = [(\bar{\varphi} \oplus \psi)|_{E_k}] = 0$ in $KK^0(E_k, D)$. For any $\varepsilon > 0$, Proposition 2.3 now implies that

$$\varphi \overset{\varepsilon/2}{\sim} W(\varphi \oplus \bar{\varphi} \oplus \psi)W^* \overset{\varepsilon/2}{\sim} \psi$$

with respect to G_k. Since G is the increasing union of the G_k, this implies that φ and ψ are approximately unitarily equivalent. ∎

We can now extend Rørdam's classification theorem (from Section 7 of [Rr1]) for direct limits of even Cuntz algebras. For simplicity, we consider only the case of simple C^*-algebras. We do have to make one modification in his setup. Every pair (G, g), in which G is a cyclic group of odd order and g is an element of G, occurs as $(K_0(M_k(\mathcal{O}_m)), [1_{M_k(\mathcal{O}_m)}])$ for some k and some even m. However, the pair $(\mathbf{Z}, 0)$ does not occur as $(K_0(M_k(\mathcal{O}_\infty)), [1_{M_k(\mathcal{O}_\infty)}])$ for any k. Therefore we will have to allow corners as well as matrix algebras. Since $\mathcal{O}_\infty$ is purely infinite and simple, every finite matrix algebra is in fact isomorphic to some corner. To simplify the statements of the results, we will therefore not consider matrix algebras over $\mathcal{O}_\infty$.

3.4. Theorem Each nonzero corner $p\mathcal{O}_\infty p$ of $\mathcal{O}_\infty$ is in the classifiable class $\mathcal{C}$ of Definition 5.1 of [ER].

Proof: Let D be a purely infinite simple C^*-algebra. In the notation of [ER], $H(p\mathcal{O}_\infty p, D)$ is the group of approximate unitary equivalence classes of nonzero homomorphisms from $p\mathcal{O}_\infty p \otimes \mathcal{K}$ to $D \otimes \mathcal{K}$ and $KL(p\mathcal{O}_\infty p, D)$ is a certain quotient of $KK^0(p\mathcal{O}_\infty p, D)$. We have to prove that the homomorphism from $H(p\mathcal{O}_\infty p, D)$ to $KL(p\mathcal{O}_\infty p, D)$ is bijective. The group $KL(p\mathcal{O}_\infty p, D)$ is defined after Lemma 5.3 in [Rr3], and in the case at hand is just $KK^0(p\mathcal{O}_\infty p, D)$, since the Ext terms in the Universal Coefficient Theorem are zero. Since $p\mathcal{O}_\infty p \otimes \mathcal{K} \cong \mathcal{O}_\infty \otimes \mathcal{K}$ and $KL(p\mathcal{O}_\infty p, D) \cong KL(\mathcal{O}_\infty, D)$, we may assume $p = 1$.

Let $\{e_{ij} : 1 \le i, j < \infty\}$ be a complete system of matrix units for $\mathcal{K}$. We have to prove that for any $\eta \in K_0(D) \cong KK^0(\mathcal{O}_\infty, D)$, there is up to approximate unitary equivalence exactly one nonzero homomorphism $\varphi : \mathcal{O}_\infty \otimes \mathcal{K} \to D \otimes \mathcal{K}$ such that $[\varphi(1 \otimes e_{11})] = \eta$ in $K_0(D)$.

For existence, choose a nonzero projection $p \in D$ such that $[p] = \eta$. Choose a proper isometry $v_1 \in pDp$, then choose an isometry $v_2 \in pDp$ whose range projection is a proper subprojection of $p - v_1 v_1^*$, an isometry $v_3 \in pDp$ whose range projection is a proper subprojection of $p - v_1 v_1^* - v_2 v_2^*$, etc., by induction. Define $\varphi_0 : \mathcal{O}_\infty \to D$ by $\varphi_0(t_j) = v_j$, and take $\varphi = \varphi_0 \otimes \mathrm{id}_{\mathcal{K}}$.

For uniqueness, let $\varphi, \psi : \mathcal{O}_\infty \otimes \mathcal{K} \to D \otimes \mathcal{K}$ be nonzero homomorphisms with the same class in KK-theory. Identify $M_n \subset \mathcal{K}$ with

$$(e_{11} + \cdots + e_{nn})\mathcal{K}(e_{11} + \cdots + e_{nn}).$$

It suffices to prove that for each n, the restrictions of φ and ψ to the corner $\mathcal{O}_\infty \otimes M_n$ are approximately unitarily equivalent. Now $\varphi(1 \otimes \sum_{i=1}^n e_{ii})$ and $\psi(1 \otimes$

$\sum_{i=1}^{n} e_{ii})$ have the same class in $K_0(D)$, so are unitarily equivalent. Therefore we may assume they are equal. Also, $\varphi(1 \otimes e_{11})$ and $\psi(1 \otimes e_{11})$ have the same class in $K_0(D)$, so there is $v_0 \in D$ such that

$$v_0 v_0^* = \varphi(1 \otimes e_{11}) \quad \text{and} \quad v_0^* v_0 = \psi(1 \otimes e_{11}).$$

Define $v \in U((D \otimes \mathcal{K})^+)$ by

$$v = 1 - \varphi\left(1 \otimes \sum_{i=1}^{n} e_{ii}\right) + \sum_{i=1}^{n} \varphi(1 \otimes e_{i1}) v_0 \psi(1 \otimes e_{1i}).$$

Then

$$v\psi(1 \otimes e_{ij})v^* = \varphi(1 \otimes e_{ij})$$

for $1 \leq i, j \leq n$. Therefore, without loss of generality, we may assume that

$$\psi(1 \otimes e_{ij}) = \varphi(1 \otimes e_{ij})$$

for $1 \leq i, j \leq n$. Now it suffices to prove that $\varphi|_{\mathcal{O}_\infty \otimes \mathbf{C} e_{11}}$ is approximately unitarily equivalent to $\psi|_{\mathcal{O}_\infty \otimes \mathbf{C} e_{11}}$, as homomorphisms from $\mathcal{O}_\infty$ to $\varphi(1 \otimes e_{11}) D \varphi(1 \otimes e_{11})$. This follows from Theorem 3.3. ∎

3.5. Theorem Let $A = \varinjlim A_n$ and $B = \varinjlim B_n$ be two simple direct limits, in which each A_n and each B_n is a finite direct sum of matrix algebras over even Cuntz algebras $\mathcal{O}_{2k}$ and corners in $\mathcal{O}_\infty$.

(1) Suppose that A and B are unital, and that there is an isomorphism

$$\alpha : (K_0(A), [1_A]) \to (K_0(B), [1_B]).$$

Then there is an isomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha$.

(2) Suppose that A and B are nonunital, and that there is an isomorphism

$$\alpha : K_0(A) \to K_0(B).$$

Then there is an isomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha$.

Proof: The previous theorem, combined with Theorem 5.9 of [Rr3], shows that these direct limits are in the class $\mathcal{C}$ of [ER]. (See the remarks after Definition 5.1 of [ER].) The result now follows from Theorem 5.7 of [Rr3]. ∎

3.6 Corollary. If p, $q \in \mathcal{O}_\infty$ are nonzero projections satisfying $[p] = \pm[q]$ in $K_0(\mathcal{O}_\infty)$, then $p\mathcal{O}_\infty p \cong q\mathcal{O}_\infty q$. In particular,

$$(1 - t_1 t_1^* - t_2 t_2^*)\mathcal{O}_\infty(1 - t_1 t_1^* - t_2 t_2^*) \cong \mathcal{O}_\infty.$$

∎

This result is of course easy if $[p] = [q]$, but seems to be new in the case $[p] = -[q]$.

3.7 Lemma Let $A = \bigoplus_{i=1}^{m} A_i$ and $B = \bigoplus_{i=1}^{n} B_i$ be finite direct sums, in which each A_i and each B_i is a finite matrix algebra over an even Cuntz algebra $\mathcal{O}_{2k}$ (with k depending on i) or a corner in $\mathcal{O}_\infty$. Let $\omega : K_0(A) \to K_0(B)$ be a homomorphism such that $\omega([1_A]) = [1_B]$. Then there is a unital homomorphism $\varphi : A \to B$ such that $\varphi_* = \omega$ and such that each partial map $\varphi_{ij} : A_i \to B_j$ is nonzero.

Proof: This is essentially done in the proof of Theorem 2.6 of [Rr1]. We need only one additional fact, namely that if $p\mathcal{O}_\infty p$ is a nonzero corner in $\mathcal{O}_\infty$, if D is a purely infinite simple C^*-algebra, and if $\omega : K_0(\mathcal{O}_\infty) \to K_0(D)$ is a homomorphism such that $\omega([p]) = [1_D]$, then there exists a unital homomorphism $\varphi : p\mathcal{O}_\infty p \to D$. (Recall that $K_0(\mathcal{O}_\infty) \cong \mathbf{Z}$, generated by [1], so that necessarily $\varphi_* = \omega$.) Choose a projection $q \in D$ such that $[1_D \oplus q] = \omega([1_{\mathcal{O}_\infty}])$ in $K_0(D)$, with $q = 0$ if $p = 1$ and $q \neq 0$ otherwise. Construct a unital homomorphism $\psi : \mathcal{O}_\infty \to (1 \oplus q)M_2(D)(1 \oplus q)$, as in the existence part of the proof of Theorem 3.4. In $K_0(D)$, we then have $[1_D] = \omega([p]) = [\psi_*(p)]$ (because $\omega([1_{\mathcal{O}_\infty}]) = [\psi(1_{\mathcal{O}_\infty})]$). Therefore there is a unitary $u \in (1 \oplus q)M_2(D)(1 \oplus q)$ such that $u\psi(p)u^* = 1 \oplus 0$. Now take $\varphi = u\psi(-)u^*|_{p\mathcal{O}_\infty p}$, regarded as a homomorphism from $p\mathcal{O}_\infty p$ to D. ∎

3.8. Theorem Let G be a countable abelian group with no odd torsion.

(1) Let $g \in G$. Then there is a unital simple C^*-algebra A, a direct limit of finite direct sums of matrix algebras over even Cuntz algebras $\mathcal{O}_{2k}$ and corners of $\mathcal{O}_\infty$ as in Theorem 3.5, such that $(K_0(A), [1]) \cong (G, g)$.

(2) There is a simple C^*-algebra A as in (1), except nonunital, such that $K_0(A) \cong G$.

Proof: We prove only the unital case. Write $G = \bigcup_{n=1}^{\infty} G_n$, where each G_n is a finitely generated subgroup of G, and $g \in G_1 \subset G_2 \subset \cdots$. Each G_n is a finite direct sum of cyclic subgroups of odd or infinite order. Therefore there is a finite direct sum $A_n = \bigoplus_{i=1}^{r(n)} A_{n,i}$, in which each $A_{n,i}$ either has the form $M_{k(n,i)}(\mathcal{O}_{m(n,i)})$ with $m(n,i)$ even, or has the form $p_{n,i}\mathcal{O}_\infty p_{n,i}$, with $p_{n,i} \in \mathcal{O}_\infty$ a nonzero projection, such that $K_0(A_n) \cong G_n$. With suitable choices of $k(n,i)$ and $p_{n,i}$, we can arrange that this isomorphism sends $[1_{A_n}]$ to g. The previous lemma provides unital homomorphisms $\varphi_n : A_n \to A_{n+1}$, with all partial maps $A_{n,i} \to A_{n+1,j}$ nonzero, such that the isomorphisms $K_0(A_n) \cong G_n$ and $K_0(A_{n+1}) \cong G_{n+1}$ identify $(\varphi_n)_*$ with the inclusion of G_n in G_{n+1}. Now set $A = \varinjlim A_n$. The nontriviality of the partial embeddings at each stage implies that A is simple. This is the desired algebra. ∎

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